

# A natural sequent calculus for Lewis' counterfactual logic $\nabla$

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## Résumé

La logique  $\nabla$  est la logique de contre-factuels la plus générale dans la famille des systèmes de Lewis. Elle est caractérisée par la classe de modèles à sphère. Dans cet article, nous proposons un nouveau calcul des séquents pour cette logique. Notre calcul est formulé en utilisant le connecteur de plausibilité comparative  $\preceq$  introduit par Lewis : une formule  $A \preceq B$  signifie intuitivement que  $A$  est au moins aussi plausible que  $B$ , de sorte qu'un conditionnel  $A \Rightarrow B$  peut être défini comme  $A$  est impossible ou  $A \wedge \neg B$  est moins plausible que  $A$ . Différemment des tentatives précédents, notre calcul est « standard » dans le sens que chaque connecteur est traité par un nombre fini de règles avec un nombre fixe et limité des prémisses. De plus, notre calcul est “interne”, dans le sens que chaque séquent peut être directement traduit dans une formule du langage. La caractéristique de notre calcul est que les séquents contiennent un type particulier de structures, appelées *blocks*, qui représentent ou codent une combinaison finie de formules avec  $\preceq$ . Nous montrons que le calcul est terminant, en conséquence il fournit une procédure de décision pour la logique  $\nabla$ .

## Abstract

The logic  $\nabla$  is the basic logic of counterfactuals in the family of Lewis' systems. It is characterized by the whole class of so-called sphere models. We propose a new sequent calculus for this logic. Our calculus takes as primitive Lewis' connective of comparative plausibility  $\preceq$  : a formula  $A \preceq B$  intuitively means that  $A$  is at least as plausible as  $B$ , so that a conditional  $A \Rightarrow B$  can be defined as  $A$  is impossible or  $A \wedge \neg B$  is less plausible than  $A$ . As a difference with previous attempts,

our calculus is standard in the sense that each connective is handled by a finite number of rules with a fixed and finite number of premises. Moreover our calculus is “internal”, in the sense that each sequent can be directly translated into a formula of the language. The peculiarity of our calculus is that sequents contain a special kind of structures, called *blocks*, which encode a finite combination of  $\preceq$ . We show that the calculus is terminating, whence it provides a decision procedure for the logic  $\nabla$ .

## 1 Introduction

In the recent history of conditional logics the work by Lewis [16] has a prominent place (among others [5, 18, 13, 11]), he proposed a formalization of conditional logics in order to represent a kind of hypothetical reasoning (if  $A$  were the case then  $B$ ), that cannot be captured by classical logic with material implication. More precisely, the original motivation by Lewis was to formalize *counterfactual sentences*, i.e. conditionals of the form “if  $A$  were the case then  $B$  would be the case”, where  $A$  is false. But independently from counterfactual reasoning, conditional logics have found then an interest also in several fields of artificial intelligence and knowledge representation. Just to mention a few : they have been used to reason about prototypical properties [8] and to model belief change [11, 9]. Moreover, conditional logics can provide an axiomatic foundation of nonmonotonic reasoning [4, 12], here a conditional  $A \Rightarrow B$  is read as “in normal circumstances if  $A$  then  $B$ ”. Finally, a kind of (multi)-conditional logics [2, 3] have been used to formalize epistemic change in a multi-agent setting and in

some kind of epistemic “games”, each conditional operator expresses the “conditional beliefs” of an agent.

In this paper we concentrate on the logic  $\forall$  of counterfactual reasoning studied by Lewis. This logic is characterized by possible world models structured by a system of spheres. Intuitively, each world is equipped with a set of nested sets of worlds : inner sets represent “most plausible worlds” from the point of view of the given world and worlds belonging only to outer sets represent less plausible worlds. In other words, each sphere represent a degree of plausibility. The (rough) intuition involving the truth condition of a counterfactual  $A \Rightarrow B$  at a world  $x$  is that  $B$  is true at the most plausible worlds where  $A$  is true, whenever there are worlds satisfying  $A$ . But Lewis is reluctant to assume that *most plausible worlds*  $A$  exist (whenever there are  $A$ -worlds), for philosophical reasons. He calls this assumption the Limit Assumption and he formulates his semantics in more general terms which do need this assumption (see below). The sphere semantics is the strongest semantics for conditional logics, in the sense that it characterizes only a subset of relatively strong systems ; there are weaker (and more abstract) semantics such as the selection function semantics which characterize a wider range of systems [18].

From the point of view of proof-theory and automated deduction, conditional logics do not have a state of the art comparable with, say, the one of modal logics, where there are well-established alternative calculi, whose proof-theoretical and computational properties are well-understood. This is partially due to the mentioned lack of a unifying semantics. Similarly to modal logics and other extensions/alternative to classical logics two types of calculi have been studied : *external* calculi which make use of labels and relations on them to import the semantics into the syntax, and *internal* calculi which stay within the language, so that a “configuration” (sequent, tableaux node...) can be directly interpreted as a formula of the language. Limiting our account to Lewis’ counterfactual logics, some external calculi have been proposed in [10] which presents modular labeled calculi for preferential logic PCL and its extensions, this family includes all counterfactual logics by Lewis. Internal calculi have been proposed by Gent [7] and by de Swart [6] for Lewis’ logic  $\forall\mathbb{C}$  and neighbours. These calculi manipulate sets of formulas and provide a decision procedure, although they comprise an *infinite set of rules and rules with a variable number of premises*. Finally in [15] the authors provide internal calculi for Lewis’ conditional logic  $\forall$  and some extensions. Their calculi are formulated for a language comprising the comparative plausibility connective, the strong and the weak conditional operator. Both conditional operators can be defined in terms of the *comparative similarity* connective. These calculi are actually an extension of Gent’s and de Swart’s ones and they comprise an infinite set of rules with a variable number of premises. We mention also a seminal work by Lamarre [13] who pro-

posed a tableaux calculus for Lewis’ logic, but it is actually a model building procedure rather than a calculus made of deductive rules.

In this paper we tackle the problem of providing a standard proof-theory for Lewis’ logic  $\forall$  in the form of internal calculi. By “standard” we mean that we aim to obtain analytic sequent calculi where each connective is handled by a finite number of rules with a fixed and finite number of premises. As a preliminary result, we propose a new internal calculus for Lewis’ logic  $\forall$ . This is the most general logic of Lewis’ family and it is complete with respect the whole class of sphere models (moreover, its unnested fragment essentially coincide with KLM rational logic  $\mathbf{R}$  [14]). Our calculus takes as primitive Lewis’ comparative plausibility connective  $\preceq$  : a formula  $A \preceq B$  means, intuitively, that  $A$  is at least as plausible as  $B$ , so that a conditional  $A \Rightarrow B$  can be defined as  $A$  is impossible or  $A \wedge \neg B$  is less plausible than  $A$ <sup>1</sup>. As a difference with previous attempts, our calculus comprises *structured* sequents containing *blocks*, where a block is a new syntactic structure encoding a finite combination of  $\preceq$ . In other words, we introduce a new modal operator (but still definable in the logic) which encodes finite combinations of  $\preceq$ . This is the main ingredient to obtain a standard and internal calculus for  $\forall$ . We show that the calculus is terminating whence it provides a decision procedure. In further research we shall study its complexity and we shall study how to extend it to stronger logics of Lewis’ family.

## 2 Lewis’ logic $\forall$

We consider a propositional language  $\mathcal{L}$  generated from a set of propositional variables and boolean connectives plus two special connectives  $\preceq$  (comparative plausibility) and  $\Rightarrow$  (conditional). A formula  $A \preceq B$  is read as “ $A$  is at least as plausible as  $B$ ”. The semantics is defined in terms of sphere models, we take the definition by Lewis without the limit assumption.

**Definition 1** *A model  $\mathcal{M}$  has the form  $\langle W, \$, [] \rangle$ , where  $W$  is a non-empty set whose elements are called worlds,  $[\ ] : Var_{prop} \rightarrow Pow(W)$  is the propositional evaluation, and  $\$ : W \rightarrow Pow(Pow(W))$ . We write  $\$_x$  for the value of the function  $\$$  for  $x \in W$ , and we denote the elements of  $\$_x$  by  $\alpha, \beta, \dots$ . Models have the following property :*

$$\forall \alpha, \beta \in \$_x \quad \alpha \subseteq \beta \vee \beta \subseteq \alpha$$

*The truth definition is the usual one for boolean cases, for the additional connectives we have :*

1. This definition avoids the Limit Assumption, in the sense that it works also for models where *at least* a sphere containing  $A$  worlds does not necessarily exist.

- $x \in [A \preceq B]$  iff  $\forall \alpha \in \mathbb{S}_x$  if  $\alpha \cap [B] \neq \emptyset$  then  $\alpha \cap [A] \neq \emptyset$
- $x \in [A \Rightarrow B]$  iff either  $\forall \alpha \in \mathbb{S}_x$   $\alpha \cap [A] = \emptyset$  or there is  $\alpha \in \mathbb{S}_x$ , such that  $\alpha \cap [A] \neq \emptyset$  and  $\alpha \cap [A \wedge \neg B] = \emptyset$ .

The semantic notions, satisfiability and validity are defined as usual.

For the ease of reading we introduce the following conventions and abbreviations : we write  $x \models A$ , where the model is understood instead of  $x \in [A]$ . Moreover given  $\alpha \in \mathbb{S}_x$ , we use the following notations :

- $\alpha \models^\forall A$  if  $\alpha \subseteq [A]$ , i.e.  $\forall y \in \alpha$   $y \models A$
- $\alpha \models^\exists A$  if  $\alpha \cap [A] \neq \emptyset$ , i.e.  $\exists y \in \alpha$  such that  $y \models A$

Observe that with this notation, the truths conditions for  $\preceq$  and  $\Rightarrow$  become :

- $x \models A \preceq B$  iff  $\forall \alpha \in \mathbb{S}_x$  either  $\alpha \models^\forall \neg B$  or  $\alpha \models^\exists A$
- $x \models A \Rightarrow B$  iff  $\forall \alpha \in \mathbb{S}_x$  either  $\alpha \models^\forall \neg A$  or there is  $\beta \in \mathbb{S}_x$ , such that  $\beta \models^\exists A$  and  $\beta \models^\forall A \rightarrow B$ .

It can be observed that the two connectives  $\preceq$  and  $\Rightarrow$  are interdefinable, in particular :

$$A \Rightarrow B \equiv (\perp \preceq A) \vee \neg(A \wedge \neg B \preceq A)$$

Also the  $\preceq$  connective can be defined in terms of the conditional  $\Rightarrow$  as follows :

$$A \preceq B \equiv (A \vee B) \Rightarrow \perp \vee \neg((A \vee B) \Rightarrow \neg A)$$

A formula of the form  $\perp \preceq \neg A$  means  $\Box A$ .

The logic  $\mathbb{V}$  can be axiomatized taking as primitive the conditional operator  $\Rightarrow$  which gives the axiomatization here below [16] :

- classical axioms and rules
- if  $A \leftrightarrow B$  then  $(C \Rightarrow A) \leftrightarrow (C \Rightarrow B)$  (RCEC)
- if  $A \rightarrow B$  then  $(C \Rightarrow A) \rightarrow (C \Rightarrow B)$  (RCK)
- $((A \Rightarrow B) \wedge (A \Rightarrow C)) \rightarrow (A \Rightarrow B \wedge C)$  (AND)
- $A \Rightarrow A$  (ID)
- $((A \Rightarrow B) \wedge (A \Rightarrow C)) \rightarrow (A \wedge B \Rightarrow C)$  (CM)
- $(A \wedge B \Rightarrow C) \rightarrow ((A \Rightarrow B) \rightarrow (A \Rightarrow C))$  (RT)<sup>2</sup>
- $((A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C)) \rightarrow ((A \wedge C) \Rightarrow B)$  (CV)
- $((A \Rightarrow C) \wedge (B \Rightarrow C)) \rightarrow (A \vee B \Rightarrow C)$  (OR)

together with the definition of  $\preceq$  in terms of  $\Rightarrow$  given above. From the axiomatization above, it can be shown that the flat fragment (i.e. without nested conditionals) of  $\mathbb{V}$  corresponds to *rational* logic **R** introduced by Kraus, Lehmann and Magidor in [12].

On the other hand, we can axiomatize  $\mathbb{V}$  taking as primitive the connective  $\preceq$  and the axioms are the following :

2. It is worth noticing that (CM) + (RT) are equivalent (in CK+ID) to the axiom known as (CSO) :

$$((A \Rightarrow B) \wedge (B \Rightarrow A)) \rightarrow ((A \Rightarrow C) \leftrightarrow (B \Rightarrow C)) \quad (CSO)$$

- classical axioms and rules
- if  $B \rightarrow (A_1 \vee \dots \vee A_n)$  then  $(A_1 \preceq B) \vee \dots \vee (A_n \preceq B)$
- $(A \preceq B) \vee (B \preceq A)$
- $(A \preceq B) \wedge (B \preceq C) \rightarrow (A \preceq C)$
- $A \Rightarrow B \equiv (\perp \preceq A) \vee \neg(A \wedge \neg B \preceq A)$

The family of Lewis' systems contains stronger logics satisfying additional axioms like *centering*, *strong centering*, *uniformity* [16], corresponding to further properties of sphere systems.

### 3 An internal sequent calculus for $\mathbb{V}$

In this section we present  $\mathcal{I}^V$ , a structured calculus for Lewis' conditional logic  $\mathbb{V}$  introduced in the previous section. In addition to ordinary formulas, sequents contains also *blocks* of the form :

$$[A_1, \dots, A_m \triangleleft B_1, \dots, B_n]$$

where each  $A_i, B_j$  are formulas. The interpretation is the following :

$$x \models [A_1, \dots, A_m \triangleleft B_1, \dots, B_n]$$

iff  $\forall \alpha \in \mathbb{S}_x$  :

- either  $\alpha \models^\forall \neg B_j$  for some  $j$ , or
- $\alpha \models^\exists A_i$  for some  $i$ .

Observe that

$$[A_1, \dots, A_m \triangleleft B_1, \dots, B_n] \leftrightarrow \bigvee_i^m \bigvee_j^n (A_i \preceq B_j)$$

Therefore a block represents  $n \times m$  disjunction of  $\preceq$  formulas.

We shall abbreviate multisets of formulas in blocks by  $\Sigma$ ,  $\Pi$ , so that we shall write (since the order is irrelevant) :

$[\Sigma \triangleleft \Pi]$ ,  $[\Sigma, A \triangleleft \Pi]$ ,  $[\Sigma \triangleleft \Pi, B]$  and so on.

A *sequent*  $\Gamma$  is a multiset  $G_1, \dots, G_k$ , where each  $G_i$  is either a formula or a block. A sequent  $\Gamma = G_1, \dots, G_k$ , is valid if for every model  $\mathcal{M} = \langle W, \mathbb{S}, [] \rangle$ , for every world  $x \in W$ , it holds that  $x \models G_1 \vee \dots \vee G_k$ . The calculus  $\mathcal{I}^V$  comprises the following axiom and rules :

- Standard Axioms :

$$(i) \Gamma, \top \quad (ii) \Gamma, \neg \perp \quad (iii) \Gamma, P, \neg P$$

- Standard external rules of sequent calculi for boolean connectives
- $(\preceq^+)$

$$\frac{\Gamma, [A \triangleleft B]}{\Gamma, A \preceq B} (\preceq^+)$$

- $(\preceq^-)$

$$\frac{\Gamma, \neg(A \preceq B), [B, \Sigma \triangleleft \Pi] \quad \Gamma, \neg(A \preceq B), [\Sigma \triangleleft \Pi, A]}{\Gamma, \neg(A \preceq B), [\Sigma \triangleleft \Pi]} (\preceq^-)$$

- $(\Rightarrow^+)$

$$\frac{\Gamma, [\perp \triangleleft A], \neg(A \wedge \neg B \preceq A)}{\Gamma, A \Rightarrow B} (\Rightarrow^+)$$

$$\text{--- } (\Rightarrow^-) \quad \frac{\Gamma, \neg(\perp \preceq A) \quad \Gamma, [A \wedge \neg B \triangleleft A]}{\Gamma, \neg(A \Rightarrow B)} \quad (\Rightarrow^-)$$

$$\text{--- (Communication)} \quad \frac{\Gamma, [\Sigma_1 \triangleleft \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \triangleleft \Pi_2] \quad \Gamma, [\Sigma_2 \triangleleft \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \triangleleft \Pi_1]}{\Gamma, [\Sigma_1 \triangleleft \Pi_1], [\Sigma_2 \triangleleft \Pi_2]} \quad (\text{Com})$$

$$\text{--- (Jump)} \quad \frac{\neg B_i, \Sigma}{\Gamma, [\Sigma \triangleleft B_1, \dots, B_n]} \quad (\text{Jump})$$

Some remark on the rules : the rule  $(\preceq+)$  just introduces the block structure, showing that  $\triangleleft$  is a generalization of  $\preceq$  ;  $(\preceq-)$  prescribes case analysis and contribute to expand the blocks ; the rules  $(\Rightarrow+)$  and  $(\Rightarrow-)$  just apply the definition of  $\Rightarrow$  in terms of  $\preceq$ . The (Communication) rule is directly motivated by the *nesting* of spheres, which means a *linear order* on sphere inclusion ; this rule is very similar to the homonymous one used in *hypersequent* calculi for handling truth in linearly ordered structures [1, 17].

As usual, given a formula  $G \in \mathcal{L}$ , in order to check whether  $G$  is valid we look for a derivation of  $G$  in the calculus  $\mathcal{T}^V$ . Given a sequent  $\Gamma$ , we say that  $\Gamma$  is derivable in  $\mathcal{T}^V$  if it admits a *derivation*. A derivation of  $\Gamma$  is a tree where :

- the root is  $\Gamma$  ;
- a leaf is an instance of standard axioms ;
- a non-leaf node is (an instance of) the conclusion of a rule having (an instance of) the premises of the rule as parents.

Here below we show a few examples of derivations.

**Example 2** A derivation of  $(A \preceq B) \vee (B \preceq A)$

$$\frac{\frac{\frac{\neg A, A}{[A \triangleleft B, A], [A, B \triangleleft A]} \quad (\text{Jump}) \quad \frac{\neg B, B}{[B \triangleleft B, A], [A, B \triangleleft B]} \quad (\text{Jump})}{\frac{[A \triangleleft B], [B \triangleleft A]}{[A \triangleleft B], B \preceq A} \quad (\preceq+) \quad \frac{[A \triangleleft B], B \preceq A}{A \preceq B, B \preceq A} \quad (\preceq+)}{\frac{A \preceq B, B \preceq A}{(A \preceq B) \vee (B \preceq A)} \quad (\vee+)} \quad (\text{Com})$$

We terminate this section by proving the soundness of the calculus and by stating some standard structural properties of the calculus.

**Theorem 4 (Soundness)** *Given a sequent  $\Gamma$ , if  $\Gamma$  is derivable then it is valid.*

**Proof:** by induction on the height of derivation. The basic case corresponds to proofs where  $\Gamma$  is an instance of standard axioms, is easy and left to the reader. For the inductive step, we have to consider all the rules ending a derivation. We only show the most interesting cases of the derivation ended by an application of  $(\preceq^-)$  and (Com) as follows :

—  $(\preceq^-)$

$$\frac{(i) \Gamma', \neg(A \preceq B), [B, \Sigma \triangleleft \Pi] \quad (ii) \Gamma', \neg(A \preceq B), [\Sigma \triangleleft \Pi, A]}{\Gamma', \neg(A \preceq B), [\Sigma \triangleleft \Pi]} \quad (\preceq^-)$$

By inductive hypothesis, (i) and (ii) are valid sequents.

By absurd, suppose that the conclusion is not, that is to say there is a model  $\mathcal{M} = \langle W, \$, [] \rangle$  and a world  $x \in W$  such that (1)  $x \not\models G_i$ , for all  $G_i \in \Gamma'$ , (2)  $x \not\models \neg(A \preceq B)$  and (3)  $x \not\models [\Sigma \triangleleft \Pi]$ . From (1), (2) and the fact that (i) is valid, we conclude that (a)  $x \models [B, \Sigma \triangleleft \Pi]$ . Reasoning in the same way, from (1), (2) and the validity of (ii), we conclude that (b)  $x \models [\Sigma \triangleleft \Pi, A]$ . By the interpretation of a block, for all  $\alpha \in \$_x$ , from (a) we have that either  $\alpha \models^\forall \neg B_j$  for some  $B_j \in \Pi$  or  $\alpha \models^\exists A_i$  for some  $A_i \in \Sigma$  or (\*)  $\alpha \models^\exists B$ . Similarly, from (b) we have that either  $\alpha \models^\forall \neg B_j$  for some  $B_j \in \Pi$  or (\*\*)  $\alpha \models^\forall \neg A$  or  $\alpha \models^\exists A_i$  for some  $A_i \in \Sigma$ . If  $\alpha \models^\forall \neg B_j$  for some  $B_j \in \Pi$ , then, by the interpretation of a block, we have that  $x \models [\Sigma \triangleleft \Pi]$ , and this contradicts (3). For the same reason, it cannot be also the case that  $\alpha \models^\exists A_i$  for some  $A_i \in \Sigma$ . The only case left is when (\*)  $\alpha \models^\exists B$  and (\*\*)  $\alpha \models^\forall \neg A$ . This contradicts (2). Indeed, (2)  $x \not\models \neg(A \preceq B)$  means that  $x \models A \preceq B$ , namely, by the truth condition of  $\preceq$ , for all  $\alpha \in \$_x$  we have that either  $\alpha \models^\forall \neg B$ , and this contradicts (\*), or  $\alpha \models^\exists A$ , and this contradicts (\*\*);

— (Com)

$$\frac{(i) \Gamma', [\Sigma_1 \triangleleft \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \triangleleft \Pi_2] \quad (ii) \Gamma', [\Sigma_2 \triangleleft \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \triangleleft \Pi_1]}{\Gamma', [\Sigma_1 \triangleleft \Pi_1], [\Sigma_2 \triangleleft \Pi_2]} \quad (\text{Com})$$

By inductive hypothesis, (i) and (ii) are valid. Suppose the conclusion  $\Gamma', [\Sigma_1 \triangleleft \Pi_1], [\Sigma_2 \triangleleft \Pi_2]$  is not, namely there is a model  $\mathcal{M} = \langle W, \$, [] \rangle$  and a world  $x \in W$  such that (1)  $x \not\models G_k$  for all  $G_k \in \Gamma'$ , (2)  $x \not\models [\Sigma_1 \triangleleft \Pi_1]$  and (3)  $x \not\models [\Sigma_2 \triangleleft \Pi_2]$ . By the interpretation of blocks, from (2) it follows that there is  $\alpha \in \$_x$  such that  $\alpha \not\models^\exists A_i$ , for all  $A_i \in \Sigma_1$ , and  $\alpha \not\models^\forall \neg B_j$  for all  $B_j \in \Pi_1$ . Similarly, from (3) it follows that there is  $\beta \in \$_x$  such that  $\beta \not\models^\exists C_k$ , for all  $C_k \in \Sigma_2$ , and  $\beta \not\models^\forall \neg D_l$  for all  $D_l \in \Pi_2$ . By Definition 1, either  $\alpha \subseteq \beta$  or  $\beta \subseteq \alpha$ . If  $\alpha \subseteq \beta$ , we have also that  $\beta \not\models^\exists A_i$ , for all  $A_i \in \Sigma_1$ , and  $\beta \not\models^\forall \neg B_j$  for all  $B_j \in \Pi_1$ . Let us consider (ii) : we have that  $\beta \not\models^\exists C_k$ , for all  $C_k \in \Sigma_2$ , as well as  $\beta \not\models^\forall \neg B_j$  for all  $B_j \in \Pi_1$  and  $\beta \not\models^\forall \neg D_l$  for all  $D_l \in \Pi_2$  : by the definition of interpretation of a block, we have that (4)  $x \not\models [\Sigma_2 \triangleleft \Pi_1, \Pi_2]$ . Furthermore, since  $\beta \not\models^\exists A_i$ , for all  $A_i \in \Sigma_1$ ,  $\beta \in \$_x$  such that  $\beta \not\models^\exists C_k$ , for all  $C_k \in \Sigma_2$  and  $\beta \not\models^\forall \neg B_j$  for all  $B_j \in \Pi_1$ , then we have that (5)  $x \not\models [\Sigma_1, \Sigma_2 \triangleleft \Pi_1]$ . However, from (1),

**Example 3** A derivation of an instance of Lewis' axiom **CV**  $((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)) \rightarrow (P \wedge Q \Rightarrow R)$ .

$$\begin{array}{c}
\frac{\neg P, P, \perp}{\dots, [P, \perp \triangleleft P]} \text{ (Jump)} \quad \frac{\perp, \neg \perp}{\dots, [\perp \triangleleft P, \perp]} \text{ (Jump)} \\
\hline
(P \wedge Q) \Rightarrow R, \neg(\perp \preceq P), [\perp \triangleleft P], \neg(P \wedge \neg \neg Q \preceq P) \quad \frac{\Delta}{P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R, [P \wedge \neg R \triangleleft P]} \\
\hline
(P \wedge Q) \Rightarrow R, P \Rightarrow \neg Q, \neg(\perp \preceq P) \quad \frac{\Delta}{P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R, [P \wedge \neg R \triangleleft P]} \text{ (}\Rightarrow^- \text{)} \\
\hline
\neg(P \Rightarrow R), P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R \quad \frac{\Delta}{\neg(P \Rightarrow R), P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R} \text{ (}\neg \text{)} \\
\hline
\neg(P \Rightarrow R), \neg\neg(P \Rightarrow \neg Q), (P \wedge Q) \Rightarrow R \quad \frac{\Delta}{\neg(P \Rightarrow R), \neg\neg(P \Rightarrow \neg Q), (P \wedge Q) \Rightarrow R} \text{ (}\wedge^- \text{)} \\
\hline
\neg((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)), (P \wedge Q) \Rightarrow R \quad \frac{\Delta}{\neg((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)), (P \wedge Q) \Rightarrow R} \text{ (}\wedge^- \text{)} \\
\hline
((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)) \rightarrow ((P \wedge Q) \Rightarrow R) \quad \frac{\Delta}{((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)) \rightarrow ((P \wedge Q) \Rightarrow R)} \text{ (}\rightarrow^+ \text{)}
\end{array}$$

where  $\Delta$  is the following derivation :

$$\begin{array}{c}
\frac{\neg P, \neg Q, P, P \wedge \neg R \quad \neg P, \neg Q, Q, P \wedge \neg R}{\neg P, \neg Q, P \wedge Q, P \wedge \neg R} \text{ (}\wedge^+ \text{)} \quad \frac{P, \neg P, \neg Q, R \quad \neg R, \neg P, \neg Q, R}{P \wedge \neg R, \neg P, \neg Q, R} \text{ (}\wedge^+ \text{)} \\
\frac{\neg P, \neg Q, P \wedge Q, P \wedge \neg R}{\neg P, \neg\neg(\neg Q), P \wedge Q, P \wedge \neg R} \text{ (}\neg \text{)} \quad \frac{P \wedge \neg R, \neg(P \wedge Q), R}{P \wedge \neg R, \neg(P \wedge Q), \neg R} \text{ (}\neg \text{)} \\
\frac{\neg P, \neg\neg(\neg Q), P \wedge Q, P \wedge \neg R}{\neg(P \wedge \neg \neg Q), P \wedge Q, P \wedge \neg R} \text{ (}\wedge^- \text{)} \quad \frac{P \wedge \neg R, \neg(P \wedge Q), \neg R}{P \wedge \neg R, \neg(P \wedge Q \wedge \neg R)} \text{ (}\wedge^- \text{)} \\
\frac{\dots, [P \wedge Q, P \wedge \neg R \triangleleft P, P \wedge \neg \neg Q]}{\dots, [P \wedge \neg R \triangleleft P, P \wedge \neg \neg Q, P \wedge Q \wedge \neg R]} \text{ (Jump)} \quad \frac{\dots, [P \wedge \neg R \triangleleft P, P \wedge \neg \neg Q, P \wedge Q \wedge \neg R]}{\dots, [P \wedge \neg R \triangleleft P, P \wedge \neg \neg Q, P \wedge Q \wedge \neg R]} \text{ (Jump)} \\
\frac{P, P \wedge \neg R, \neg P}{\dots, [P, P \wedge \neg R \triangleleft P]} \text{ (Jump)} \quad \frac{\dots, [P \wedge \neg R \triangleleft P, P \wedge \neg \neg Q, P \wedge Q \wedge \neg R]}{[\perp \triangleleft P], [\perp \triangleleft P \wedge Q], [P \wedge \neg R \triangleleft P, P \wedge \neg \neg Q], \neg(P \wedge Q \wedge \neg R \preceq P \wedge Q), \neg(P \wedge \neg \neg Q \preceq P)} \text{ (}\preceq^- \text{)} \\
\hline
[\perp \triangleleft P], [P \wedge \neg R \triangleleft P], [\perp \triangleleft P \wedge Q], \neg(P \wedge Q \wedge \neg R \preceq P \wedge Q), \neg(P \wedge \neg \neg Q \preceq P) \quad \frac{\Delta}{[\perp \triangleleft P], [P \wedge \neg R \triangleleft P], [\perp \triangleleft P \wedge Q], \neg(P \wedge Q \wedge \neg R \preceq P \wedge Q), \neg(P \wedge \neg \neg Q \preceq P)} \text{ (}\Rightarrow^+ \text{)} \\
\hline
[\perp \triangleleft P], \neg(P \wedge \neg \neg Q \preceq P), (P \wedge Q) \Rightarrow R, [P \wedge \neg R \triangleleft P] \quad \frac{\Delta}{[\perp \triangleleft P], \neg(P \wedge \neg \neg Q \preceq P), (P \wedge Q) \Rightarrow R, [P \wedge \neg R \triangleleft P]} \text{ (}\Rightarrow^+ \text{)} \\
\hline
P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R, [P \wedge \neg R \triangleleft P]
\end{array}$$

(4) and (5) we obtain that (ii) is not valid, against the inductive hypothesis ; If  $\beta \subseteq \alpha$ , the proof is symmetric and left to the reader. ■

**Proposition 5 (Weakening)** *Weakening is height-preserving admissible in the following cases :*

- if  $\Gamma$  is derivable, then  $\Gamma, F$  is derivable where  $F$  is a formula or a block.
- if  $\Gamma, [\Sigma \triangleleft \Pi]$  is derivable, so are  $\Gamma, [\Sigma, A \triangleleft \Pi]$  and  $\Gamma, [\Sigma \triangleleft \Pi, B]$

**Proposition 6 (Contraction)** *Contraction is height-preserving admissible in the following cases :*

- if  $\Gamma, [A, A, \Sigma \triangleleft \Pi]$  is derivable then  $\Gamma, [A, \Sigma \triangleleft \Pi]$  is derivable too.
- if  $\Gamma, [\Sigma \triangleleft \Pi, B, B]$  is derivable then  $\Gamma, [\Sigma \triangleleft \Pi, B]$  is derivable too.
- if  $\Gamma, F, F$  is derivable then  $\Gamma, F$  is derivable too, where  $F$  is either a formula or a block.

**Proposition 7 (Invertibility)** *All rules, except (Jump), are height-preserving invertible : if the conclusion is derivable then the premises must be derivable with a derivation of no-greater height.*

## 4 Termination and Completeness

In this section we prove both the termination and the completeness of the calculus. Both results make use of the notion of *saturated sequent* : intuitively any sequent that is obtained by backwards applying the rules “as much as possible”. To get termination we show that any derivation without *redundant* application of the rules is finite and its leaves are axioms or saturated sequents. Completeness is proved by induction on the modal degree of a sequent (defined next), by taking advantage of the fact that backward application of the rules does not increase the modal degree of sequent and eventually reduced it (the Jump rule).

**Definition 8** *The modal degree md of a formula/sequent is defined as follows :*

$$\begin{aligned}
md(P) &= 0 \\
md(A * B) &= \max(md(A), md(B)), \text{ for } * \text{ boolean connective} \\
md(\neg A) &= md(A) \\
md(A \preceq B) &= md(A \Rightarrow B) = \max(md(A), md(B)) + 1 \\
md(\Delta) &= \max\{md(A) \mid A \in \Delta\} \text{ for a multiset}
\end{aligned}$$

$$\Delta$$

$$md(\Sigma \triangleleft \Pi) = \max(md(\Sigma), md(\Pi)) + 1$$

**Definition 9** A sequent  $\Gamma$  is saturated if it has the form

$$\Gamma_N, \Lambda, [\Sigma_1 \triangleleft \Pi_1], \dots, [\Sigma_n \triangleleft \Pi_n]$$

where  $\Gamma_N, \Lambda$  are possible empty and  $n \geq 0$  and :

1.  $\Gamma_N$  is a multi-set of negative  $\preceq$ -formulas,
2.  $\Lambda$  is a multi-set of literals,
3. for every  $\neg(A \preceq B) \in \Gamma_N$  and every  $[\Sigma_i \triangleleft \Pi_i]$  either  $B \in \Sigma_i$  or  $A \in \Pi_i$
4. for every  $[\Sigma_i \triangleleft \Pi_i]$  and  $[\Sigma_j \triangleleft \Pi_j]$  : either  $\Sigma_i \subseteq \Sigma_j$  or  $\Sigma_j \subseteq \Sigma_i$  and either  $\Pi_i \subseteq \Pi_j$  or  $\Pi_j \subseteq \Pi_i$ .

**Proposition 10** All rules preserve the modal degree, i.e. the premises of rules have a modal degree no-greater than the one of the respective conclusion.

We want to prove now that the calculus terminates, provided we restrict attention to *non-redundant* derivations, a notion that we define next. An application of a rule (R) :

$$\frac{\Gamma_1 \quad \Gamma_2}{\Gamma} (R)$$

is redundant if  $\Gamma$  can be obtained from  $\Gamma_i$  for  $i = 1$  or  $i = 2$  by *contraction* or *weakening*. A derivation is non-redundant if (a) it does not contain redundant applications of the rules, (b) if a sequent is an axiom then it is a leaf of the derivation. As a consequence of the admissibility of contraction (Proposition 6) and of weakening (Proposition 5), if a sequent is derivable then it has a non-redundant derivation. Thus we can safely restrict proof search to non-redundant derivations.

The proposition below means that for any sequent  $\Gamma$  (derivable or not in the calculus), there is a (non-redundant) derivation tree whose leaves (no matter whether they are derivable or not in the calculus) are saturated sequents with no greater modal degree. In order to prove it, we introduce some complexity measure of sequents. First we define a complexity measure of formulas :

$$Cp(A) = 0 \text{ if } A \text{ is either a literal or it has the form } \neg(C \preceq D),$$

$$Cp(A) = 1 \text{ if } A \text{ has one of the forms } C \preceq D, C \Rightarrow D, \neg(C \Rightarrow D)$$

$$Cp(\neg\neg A) = Cp(A) + 1$$

$$Cp(A * B) = Cp(A) + Cp(B) + 1 \text{ where } * \text{ is a boolean connective.}$$

Next we let

$$CP(\Gamma) = \text{multiset}\{Cp(A) \mid A \in \Gamma\}$$

We further define :

$$CN(\Gamma) = \text{Card}(\{\neg(A \preceq B), [\Sigma \triangleleft \Pi] \mid \neg(A \preceq B), [\Sigma \triangleleft \Pi] \in \Gamma, B \notin \Sigma, A \notin \Pi\})$$

$$CC(\Gamma) = n * (\text{Card}(\Sigma_\Gamma) + \text{Card}(\Pi_\Gamma)) - \sum_{i=1}^n (\text{Card}(\Sigma_i) + \text{Card}(\Pi_i))$$

We finally define the rank of a sequent  $\Gamma$ ,  $rank(\Gamma)$  as the triple

$$rank(\Gamma) = \langle CP(\Gamma), CN(\Gamma), CC(\Gamma) \rangle$$

taken in *lexicographic* order, where we consider the *multiset ordering* for  $CP(\Gamma)$ . Observe that a *minimal* rank has the form  $\langle 0^*, 0, m \rangle$ , where  $m \geq 0$ .

We are ready to prove the following proposition.

**Proposition 11** Given a sequent  $\Gamma$ , every branch of any derivation-tree starting with  $\Gamma$  eventually ends with a saturated sequent with no greater modal degree than that of  $\Gamma$ . Moreover the set of such saturated sequents for a given derivation tree is finite.

**Proof:** by Proposition 10, no rule applied backward augments the modal degree of a sequent. It can be shown that every (non-redundant) application of a rules (R) with premises  $\Gamma_i$  and conclusion  $\Gamma$  reduces the rank of  $\Gamma$  in the sense that  $rank(\Gamma_i) < rank(\Gamma)$ . In order to see this, we note :

- the application of classical propositional rule reduces  $CP(\Gamma)$
- the application of  $(\preceq^+), (\Rightarrow^+), (\Rightarrow^-)$  rules reduces  $CP(\Gamma)$
- the application of  $(\preceq^-)$  reduces  $CN(\Gamma)$ , *without increasing*  $CP(\Gamma)$
- the application of  $(Com)$  reduces  $CC(\Gamma)$ , *without increasing* neither  $CP(\Gamma)$ , nor  $CN(\Gamma)$ . We first show that an application of  $(Com)$  rule reduces  $CC(\Gamma)$ . Let  $\Gamma = \Delta, [\Sigma_1 \triangleleft \Pi_1], [\Sigma_2 \triangleleft \Pi_2], \dots, [\Sigma_n \triangleleft \Pi_n]$ . To simplify indexing (since the order does not matter) suppose that the application of  $(Com)$  concerns the blocks  $[\Sigma_1 \triangleleft \Pi_1], [\Sigma_2 \triangleleft \Pi_2]$ , so that from the premises of the application of  $(Com)$  leading to  $\Gamma$  will be :

$$\Gamma_1 = \Delta, [\Sigma_1 \triangleleft \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \triangleleft \Pi_2], [\Sigma_3 \triangleleft \Pi_3], \dots, [\Sigma_n \triangleleft \Pi_n]$$

$$\Gamma_2 = \Delta, [\Sigma_2 \triangleleft \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \triangleleft \Pi_1], [\Sigma_3 \triangleleft \Pi_3], \dots, [\Sigma_n \triangleleft \Pi_n]$$

Observe that the overall set of formulas in blocks does not change so that, referring to the above notation :

$$\Sigma_{\Gamma_i} = \Sigma_\Gamma \text{ and } \Pi_{\Gamma_i} = \Pi_\Gamma, \quad \text{for } i = 1, 2$$

Let us abbreviate  $a = n * (\text{Card}(\Sigma_\Gamma) + \text{Card}(\Pi_\Gamma))$  and  $c = \sum_{i=3}^n (\text{Card}(\Sigma_i) + \text{Card}(\Pi_i))$ , so that we have :

$$CC(\Gamma) = a - ((\text{Card}(\Sigma_1) + \text{Card}(\Pi_1)) + (\text{Card}(\Sigma_2) + \text{Card}(\Pi_2))) + c$$

$$CC(\Gamma_1) = a - ((Card(\Sigma_1) + Card(\Pi_1 \cup \Pi_2)) + (Card(\Sigma_1 \cup \Sigma_2) + Card(\Pi_2)) + c)$$

$$CC(\Gamma_2) = a - ((Card(\Sigma_2) + Card(\Pi_1 \cup \Pi_2)) + (Card(\Sigma_1 \cup \Sigma_2) + Card(\Pi_1)) + c)$$

Obviously  $CC(\Gamma_1) \leq CC(\Gamma)$  and  $CC(\Gamma_2) \leq CC(\Gamma)$ , since  $Card(\Sigma_1 \cup \Sigma_2) \geq Card(\Sigma_i)$  and  $Card(\Pi_1 \cup \Pi_2) \geq Card(\Pi_i)$ ,  $i = 1, 2$ . But since the application of  $(Com)$  is non-redundant, it respects the restriction  $(RestCom)$  and therefore either (a)  $\Sigma_1 \not\subseteq \Sigma_2$  and  $\Sigma_2 \not\subseteq \Sigma_1$  or (b)  $\Pi_1 \not\subseteq \Pi_2$  and  $\Pi_2 \not\subseteq \Pi_1$ . It is easy to see that some inequalities must be strict by the  $(RestCom)$  restriction, so that in both cases (a) and (b) we get  $CC(\Gamma_1) < CC(\Gamma)$  and  $CC(\Gamma_2) < CC(\Gamma)$ . ■

The following theorem shows that the calculus is terminating, whence it provides a decision procedure for  $\mathbb{V}$ , assuming restriction to non-redundant derivations.

**Proposition 12** *Given a sequent  $\Gamma$ , any non-redundant derivation-tree of  $\Gamma$  is finite.*

**Proof:** by induction on the modal degree  $m$  of  $\Gamma$ . If  $m = 0$  then we rely on the corresponding property of classical sequent calculus. If  $m > 0$ , by the previous Proposition 11,  $\Gamma$  has a finite derivation tree ending with a set of saturated sequents  $\Gamma_i$ . For each  $\Gamma_i$  either it is an axiom and  $\Gamma_i$  will be a leaf of the derivation, or the only applicable rule (by non-redundancy restriction) is  $(Jump)$ , but the premise of  $(Jump)$  has a smaller modal degree and we apply the induction hypothesis to the premise of  $(Jump)$ . ■

The following proposition is the last ingredient we need for the completeness proof.

**Proposition 13 (Semantic Invertibility)** *All rules, except  $(Jump)$  are semantically invertible : if the conclusion is valid then the premises are also valid.*

**Theorem 14 (Completeness of the calculus)** *If  $\Gamma$  is valid then it is derivable*

**Proof:** by induction on the modal degree of  $\Gamma$ . If  $md(\Gamma) = 0$  then  $\Gamma$  is just a multiset of propositional formulas, and we rely on the completeness of sequent calculus for classical logic.

Suppose now that  $md(\Gamma) > 0$ , by Proposition 11,  $\Gamma$  can be derived from a set of saturated sequents  $\Gamma_i$  of no greater modal degree. But by previous Proposition 13 (semantic invertibility) since  $\Gamma$  is valid then also each  $\Gamma_i$  is valid. We

are left to prove that any *saturated and valid* sequent  $\Gamma_i$  is derivable.

To this purpose we prove that if  $\Gamma_i$  is valid then either (i) it is an axiom or (ii) there must exist a *valid* sequent  $\Delta$  such that  $\Gamma_i$  is obtained by  $(Jump)$  from  $\Delta$ . In the first case (i) the result is obvious. In case (ii) we reason as follows : since  $md(\Delta) < md(\Gamma_i)$  by induction hypothesis  $\Delta$  is derivable in the calculus, and so is  $\Gamma_i$  indeed by the  $(Jump)$  rule.

Let us prove the fact (ii) : if  $\Gamma_i$  is valid and saturated and it is not an axiom, then there exist a *valid* sequent  $\Delta$  such that  $\Gamma_i$  is obtained by  $(Jump)$  from  $\Delta$ .

Suppose that  $\Gamma_i$  is valid and it is not an axiom. We let

$$\Gamma_i = \Gamma_N, \Lambda, [\Sigma_1 \triangleleft \Pi_1], \dots, [\Sigma_n \triangleleft \Pi_n]$$

as in the definition of saturated sequent. Observe that  $\Lambda$  does not contain axioms. By saturation (and weakening and contraction) we can assume that the blocks in the sequence are ordered as follows :

$$\begin{aligned} - \Sigma_1 \supseteq \Sigma_2 \supseteq \dots \supseteq \Sigma_n \\ - \Pi_1 \subseteq \Pi_2 \subseteq \dots \subseteq \Pi_n \end{aligned}$$

A quick argument : by saturation blocks are ordered with respect to set-inclusion for both components  $\Sigma$  and  $\Pi$ , consider them ordered first by decreasing  $\Sigma$  : let two blocks in the sequence :  $[\Sigma \triangleleft \Pi]$ ,  $[\Sigma' \triangleleft \Pi']$  with  $\Sigma' \subseteq \Sigma$  we can assume that  $\Pi \subseteq \Pi'$  otherwise it would be  $\Pi' \subseteq \Pi$ , but then any sequent containing **both**  $[\Sigma \triangleleft \Pi]$ ,  $[\Sigma' \triangleleft \Pi']$  is semantically equivalent to a sequent containing **only**  $[\Sigma \triangleleft \Pi]$  (syntactically we get rid of  $[\Sigma' \triangleleft \Pi']$  by weakening and contraction)<sup>3</sup>. Thus we let :

$$\begin{aligned} \Pi_1 &= B_{1,1}, \dots, B_{1,k_1} \\ \Pi_2 &= B_{1,1}, \dots, B_{1,k_1}, B_{2,1}, \dots, B_{2,k_2} \\ &\dots \\ \Pi_n &= B_{1,1}, \dots, B_{1,k_1}, \dots, B_{2,k_2}, \dots, B_{n,k_n} \end{aligned}$$

Suppose now by absurdity that no application of  $(Jump)$  leads to a valid sequent. Thus for each  $l = 1, \dots, n$ , and  $t = 1, \dots, k_l$ , the sequent

$$\neg B_{l,t}, \Sigma_l$$

is not valid. Starting from  $l = 1$  up to  $n$ , there are increasing sequences of models :

$$\begin{aligned} \mathcal{M}_{1,1}, \dots, \mathcal{M}_{1,k_1}, \\ \mathcal{M}_{1,1}, \dots, \mathcal{M}_{1,k_1}, \mathcal{M}_{2,1}, \dots, \mathcal{M}_{2,k_2} \\ \mathcal{M}_{1,1}, \dots, \mathcal{M}_{1,k_1}, \dots, \mathcal{M}_{2,k_2}, \dots, \mathcal{M}_{n,k_n} \end{aligned}$$

where  $\mathcal{M}_{l,t} = (W_{l,t}, \mathcal{S}^{l,t}, [ ]_{l,t})$  for  $l = 1, \dots, n$ , and  $t = 1, \dots, k_l$  and some elements  $x_{l,t} \in W_{l,t}$  such that

$$\mathcal{M}_{l,t}, x_{l,t} \models B_{l,t}$$

and

$$\mathcal{M}_{l,t}, x_{l,t} \not\models C \text{ for all } C \in \Sigma_l$$

3. An alternative argument :  $\Gamma_i$  must contain a *valid* subsequent  $\Gamma'_i$  where the blocks satisfy the above ordering conditions. Then the proof carry on considering  $\Gamma'_i$ .

Observe that if  $\mathcal{M}_{l,t}, x_{l,t} \not\models C$  for all  $C \in \Sigma_s$  and  $s < t$  then  $\mathcal{M}_{l,t}, x_{l,t} \not\models C$  for all  $C \in \Sigma_t$ , as  $\Sigma_t \subseteq \Sigma_s$ .

We suppose that all models are disjoint and we define a new model  $\mathcal{M} = \langle \mathcal{W}, \$, [ ] \rangle$  as follows :

$$\begin{aligned} \mathcal{W} &= (\bigcup_l \bigcup_t (\mathcal{W}_{l,t})) \cup \{x\} \text{ for a new element } x \\ \$_z &= \$_{z,t}^{l,t} \text{ if } z \in \mathcal{W}_{l,t} \text{ for some } l, t \\ [P] &= \bigcup_l \bigcup_t [P]_{l,t} \text{ if } \neg P \notin \Lambda \\ [P] &= \bigcup_l \bigcup_t [P]_{l,t} \cup \{x\} \text{ if } \neg P \in \Lambda \end{aligned}$$

In order to define  $I(x)$  we let :

$$\begin{aligned} \alpha_1 &= \{x_{1,1}, \dots, x_{1,k_1}\} \\ \alpha_2 &= \{x_{1,1}, \dots, x_{1,k_1}, x_{2,1}, \dots, x_{2,k_2}\} \\ &\dots \\ \alpha_n &= \{x_{1,1}, \dots, x_{1,k_1}, x_{2,1}, \dots, x_{2,k_2}, \dots, x_{n,k_n}\} \end{aligned}$$

We finally let

$$\$_x = \{\alpha_1, \dots, \alpha_n\}$$

Observe that the ‘‘spheres’’  $\alpha_l$  are nested.

To complete the proof we must show that  $x$  falsifies  $\Gamma_i$  in  $\mathcal{M}$ . In particular we have to show that :

1.  $\mathcal{M}, x \not\models L$  for every  $L \in \Lambda$
2.  $\mathcal{M}, x \not\models [\Sigma_l \triangleleft \Pi_l]$  for  $l = 1, \dots, n$
3.  $\mathcal{M}, x \not\models \neg(A \preceq B)$  for every  $\neg(A \preceq B) \in \Gamma_N$

Fact (1) is obvious by definition : if  $P \in \Lambda$ , then  $\neg P \notin \Lambda$  (otherwise  $\Gamma_i$  would be an axiom) and  $x \notin [P]$ , if  $\neg P \in \Lambda$ , then  $x \in [P]$ .

To prove Fact (2), first observe that for  $z \in \mathcal{W}_{l,t}$  and every formula  $F$ , we have

$$z \in [F] \iff z \in [F]_{l,t}$$

This is proved by a straightforward induction on  $F$ . Then we prove by induction on  $l$  Fact (2). For  $l = 1$ , we have that for  $x_{1,l} \in \alpha_1$

$$\mathcal{M}, x_{1,l} \models B_{1,l}$$

whence :

$$\alpha_1 \not\models^\forall \neg B_{1,t} \text{ for } t = 1, \dots, k_1.$$

On the other hand putting  $\Sigma_1 = C_{1,1}, \dots, C_{1,r_1}$ , we have, for every  $u = 1, \dots, r_1$  and  $x_{1,t}, t = 1, \dots, k_1$

$$\mathcal{M}, x_{1,t} \not\models C_{1,u}$$

but this means that

$$\alpha_1 \not\models^\exists C_{1,u} \text{ for } u = 1, \dots, r_1.$$

Thus we get

$$\mathcal{M}, x \not\models [\Sigma_1 \triangleleft \Pi_1]$$

For  $l > 1$ , since  $\Sigma_l \supseteq \Sigma_{l-1}$  and  $\Pi_{l-1} \subseteq \Pi_l$ , the argument is the same (using possibly the induction hypothesis).

We consider now (Fact 3) : let  $\neg(A \preceq B) \in \Gamma_N$  and let  $\alpha_l \in \$_x$ . Let us consider  $[\Sigma_l \triangleleft \Pi_l]$ , by saturation either  $A \in \Pi_l$  or  $B \in \Sigma_l$ . For what we have just shown, in the former case we have  $\alpha_l \models^\exists A$  and in the latter case we have  $\alpha_l \models^\forall \neg B$ . Thus :

for any  $\alpha_l \in \$_x$  either  $\alpha_l \models^\exists A$  or  $\alpha_l \models^\forall \neg B$ .

whence  $\mathcal{M}, x \models A \preceq B$ . ■

## 5 Conclusions

In this paper we begin a proof-theoretical investigation of Lewis’ logics of counterfactuals characterized by the sphere-model semantics. We have presented a simple, analytic calculus  $\mathcal{I}^\forall$  for logic  $\forall$ , the most general logic characterized by the sphere-model semantics. The calculus is *standard*, that is to say it contains a finite a number of rules with a fixed number of premisses and *internal* in the sense that each sequent denotes a formula of  $\forall$ . The novel ingredient of  $\mathcal{I}^\forall$  is that sequents are structured objects containing blocks, where a block is a structure or a sort of n-ary modality encoding a finite combination of formulas with the connective  $\preceq$ . The calculus  $\mathcal{I}^\forall$  ensures termination, and therefore it provides a decision procedure for  $\forall$ .

In future research, we aim at extending our approach to all the other conditional logics of the Lewis’ family, in particular we aim at focusing on the logics  $\forall\text{T}$ ,  $\forall\text{W}$  and  $\forall\text{C}$ . Moreover, we shall study the complexity of the calculus  $\mathcal{I}^\forall$  with the hope of obtaining optimal calculi.

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